Period 3, May 21, 2025

Theorem 8.1 Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = xe^x - \int e^x \cdot dx = xe^x - e^x$$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$y = xe^x - e^x$

$$\frac{dy}{dx} = \cancel{1 \cdot e^x} + x \cdot e^x - \cancel{e^x} = xe^x$$

$$\int xe^x\,dx = xe^x - e^x + C$$

Example:

$\int x^2 \sin x\,dx$

$u = x^2 \qquad dv = \sin x\,dx$

$du = 2x\,dx \qquad v = -\cos x$

$$\int x^2 \sin x\,dx = -x^2\cos x - \int -\cos x \cdot 2x\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\left[x\sin x - \int \sin x \cdot dx\right]$$

$$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x - 2(-\cos x) + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$\cancel{-2x\cos x} - x^2 \cdot -\sin x + \cancel{2\sin x} + \cancel{2x\cos x} - \cancel{2\sin x} = x^2\sin x$$

Example:

$\int arc\sin x\,dx$ $= x\arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$

$u = \arcsin x$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$u = 1-x^2$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$= x\arc\sin x - \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{-2\cancel{x}} = x\arcsin x + \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= x\arcsin x + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} \cdot u^{-\frac{1}{2}+1=\frac{1}{2}} + C$

$= x\arcsin x + \sqrt{1-x^2} + C$

Example:

$\int e^x \sin x\,dx$ $= -e^x\cos x - \int -\cos x \cdot e^x\,dx = -e^x\cos x + \int \cos x \cdot e^x\,dx$

$u = e^x$ $\quad dv = \sin x\,dx$

$du = e^x dx$ $\quad v = -\cos x$

$u = e^x$ $\quad dv = \cos x\,dx$

$du = e^x dx$ $\quad v = \sin x$

$\int e^x \sin x\,dx = -e^x\cos x + e^x\sin x - \cancel{\int \sin x \cdot e^x\,dx}$

$+\int e^x \sin x\,dx$ $\qquad + \cancel{\int e^x \sin x\,dx}$

$\frac{2\int e^x \sin x\,dx}{\cancel{2}} = \frac{-e^x\cos x + e^x\sin x}{2}$

$$\int x^3 \sin 2x\, dx = -\frac{x^3}{2}\cos 2x - \int\left(-\frac{1}{2}\cos 2x\right)\cdot 3x^2\, dx$$

$u = x^3 \qquad dv = \sin 2x\, dx$

$du = 3x^2\, dx \qquad v = -\frac{1}{2}\cos 2x$

$$= -\frac{x^3}{2}\cos 2x + \frac{3}{2}\int x^2 \cos 2x\, dx$$

$u = x^2 \qquad dv = \cos 2x$

$du = 2x\, dx \qquad v = \frac{1}{2}\sin 2x$

$$= -\frac{x^3}{2}\cos 2x + \frac{3}{2}\left[\frac{x^2}{2}\sin 2x - \int\left(\cancel{\frac{1}{2}}\sin 2x\right)\cancel{2}x\, dx\right]$$

$$= -\frac{x^3}{2}\cos 2x + \frac{3x^2}{4}\sin 2x - \frac{3}{2}\int x \sin 2x\, dx$$

$u = x \qquad dv = \sin 2x\, dx$

$du = dx \qquad v = -\frac{1}{2}\cos 2x$

$$= -\frac{x^3}{2}\cos 2x + \frac{3x^2}{4}\sin 2x - \frac{3}{2}\left[-\frac{x}{2}\cos 2x \cancel{-} \int +\frac{1}{2}\cos 2x\, dx\right]$$

$$= -\frac{x^3}{2}\cos 2x + \frac{3x^2}{4}\sin 2x + \frac{3x}{4}\cos 2x - \frac{3}{4}\int \cos 2x\, dx$$

$$= -\frac{x^3}{2}\cos 2x + \frac{3x^2}{4}\sin 2x + \frac{3x}{4}\cos 2x - \frac{3}{4}\cdot\frac{1}{2}\cdot\sin 2x + C$$

$$= -\frac{x^3}{2}\cos 2x + \frac{3x^2}{4}\sin 2x + \frac{3x}{4}\cos 2x - \frac{3}{8}\sin 2x + C$$

$$\int x^3 \sin 2x\, dx = x^3\left(-\tfrac{1}{2}\cos 2x\right) - 3x^2\left(-\tfrac{1}{4}\sin 2x\right) + 6x \cdot \tfrac{1}{8}\cos 2x - 6 \cdot \tfrac{1}{16}\sin 2x$$

	u	dv
		$\sin 2x\,dx$
$+$	x^3	$-\frac{1}{2}\cos 2x$
$-$	$3x^2$	$-\frac{1}{4}\sin 2x$
$+$	$6x$	$\frac{1}{8}\cos 2x$
$-$	6	$\frac{1}{16}\sin 2x$

$$= \frac{-x^3}{2}\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3x}{4}\cos 2x - \frac{3}{8}\sin 2x + C$$

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4. $\int x^2 \cos x\, dx$

5. $\int e^{2x} \cos x \, dx$

$u = e^{2x}$ $\quad dv = \cos x \, dx$

$du = 2e^{2x} dx$ $\quad v = \sin x$

$$\int e^{2x} \cos x \, dx = e^{2x} \cdot \sin x - \int \sin x \cdot 2e^{2x} dx = e^{2x} \sin x - 2\int e^{2x} \sin x \, dx$$

$u = e^{2x}$ $\quad dv = \sin x \, dx$

$du = 2e^{2x} dx$ $\quad v = -\cos x$

$$= e^{2x} \sin x - 2\left[-e^{2x} \cos x - \int -\cos x \cdot 2e^{2x} dx\right]$$

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x - \cancel{4\int e^{2x} \cos x \, dx}$$

$+4\int e^{2x} \cos x \, dx$ $\quad + \cancel{4\int e^{2x} \cos x \, dx}$